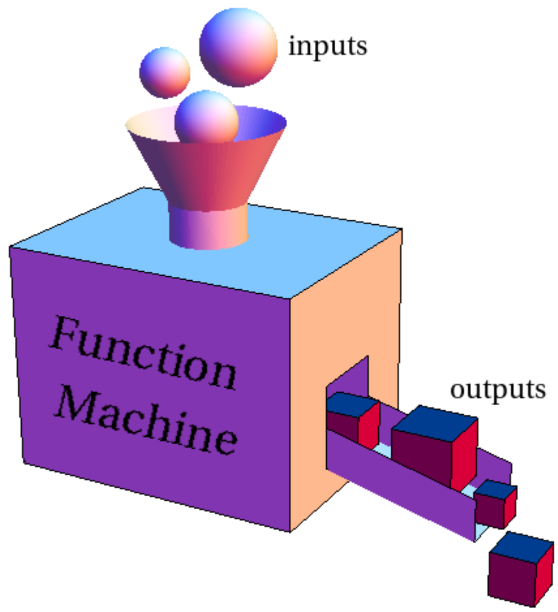
Chapter 3: FUNCTIONS



In this chapter…

"Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them"

quote by the late French mathematician Henri Poincare

Perhaps, without knowing it, we have already looked at so-called "functions" in previous sections while exploring the relationship between figures in a pattern and their corresponding quantities, such as number of tiles, perimeter etc. We will define "functions" more formally later; for now, let us note that functions are special relationships between sets of objects/numbers.

Whenever we find an explicit formula to express some quantity (based on another quantity), we establish a relationship between the two quantities. This type of relationship is called a function.

In this chapter we explore different representations of functions and their properties. First we highlight corresponding national and state standards.

According to the **National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics…**

In prekindergarten through grade 2 all students should…

- recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another
- describe qualitative change, such as a student's growing taller

In grades 3–5 all students should…

- represent and analyze patterns and functions, using words, tables, and graphs
- investigate how a change in one variable relates to a change in a second variable

In grades 6–8 all students should…

- represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules
- relate and compare different forms of representation for a relationship
- model and solve contextualized problems using various representations, such as graphs, tables, and equations

In the **Massachusetts Mathematics Curriculum Framework…**

Grade 5 - 5.OA.B. Analyze patterns and relationships.

> 3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Grade 6 - 6.EE.C. Represent and analyze quantitative relationships between dependent and independent variables.

> 9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Grade 8

8.F.A. Define, evaluate, and compare functions.

> 1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
>
> 2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

8.F.B. Use functions to model relationships between quantities.

> 4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
>
> 5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

It is worth noting that elementary school teachers should be well versed in middle school mathematics. Think about it as knowing what your students will face in the future in order to be qualified to prepare them for it. Teachers should also be able to foreshadow future mathematics for their students and answer questions about how concepts in mathematics will build upon each other.

3.1 Introduction to Functions

We begin by revisiting a problem from the chapter on patterns to see how growing patterns can be expressed as a relationship between two quantities.

Example 3.1.1

Recall the pattern below from the "Growing Patterns" section

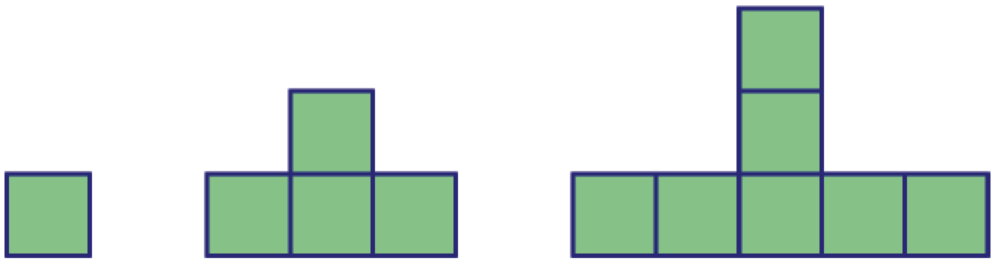


Figure 1 Figure 2 Figure 3

We found the number of tiles in the nth figure to be 1 + 3(n - 1) = 3n - 2. This explicit formula gives us a relationship between the figure number n and the number of tiles required for that figure. We can represent this relationship in many different ways as follows:

Words: The number of tiles in the nth figure is 3n - 2 for n = 1, 2, 3, …. (and so on)

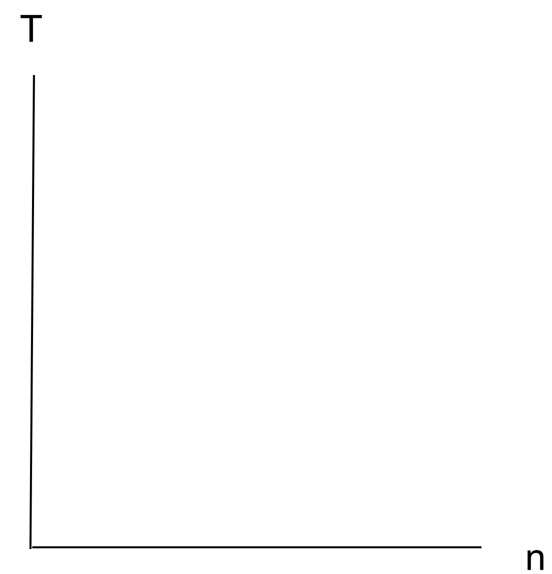
Equation: T = 3n - 2 where n is the figure number and T represents the number of tiles in figure n. This gives us a rule of how to find an output value T that corresponds to an input value n.

Table: The first column shows input values and the second column shows the corresponding output values.

Figure number (*n*) (input)	number of tiles (T) (output)
1	1
2	4
3	7
4	10
n	3n - 2

Set of ordered pairs/points: $\{(1, 1), (2, 4), (3, 7), (4, 10), \ldots (n, 3n-2), \ldots\}$
The first number in each pair gives an input value (figure number) and the second number in each pair gives the corresponding output value (number of tiles in the corresponding figure). Each pair in the set is of the form (input, output).
The braces { } notation on the outside is commonly used to enclose a set.
In this case it is a set of points or ordered pairs.

Graph: Plot points on the axes below.



The relationship described in example 3.1.1 between the figure numbers and the number of tiles required for each figure is a function (more details later) and functions are specific types of so called relations so we will define relations first.

Relations

A **relation** is a relationship between two sets of objects/numbers in which objects from one set (of inputs) are paired with objects from the other set (of outputs). The **domain** of a relation is the set of all input and the **range** of a relation is the set of all outputs.

Example 3.1.1 gives a relation between the figure number and the number of tiles required for that figure.

Example 3.1.2

What is the domain and range of our relationship from Example 3.1.1?

When studying these concepts in terms of the xy-plane (Cartesian Coordinate system), the variable x is often used to represent input (values along the horizontal axis) and y is often used to represent output (values along the vertical axis).

In particular, in Example 3.1.1, if we use x in place of n and y in place of T, the equation $T = 3n - 2$ becomes $y = 3x - 2$. Observe this is a *linear relationship* (the graph forms a line - if we connect the points). In the next chapter we will study linear functions in depth.

This connection enables us to see the importance of studying patterns in elementary and middle school and its relationship to the study of functions in high school. When you teach your future students about patterns, you will be giving them the preparation to study functions which is the foundation of mathematical modeling!

Input/output tables are used in elementary and middle school to give students an understanding of the relationship between two variables.

Input/output video with single operations: https://www.youtube.com/watch?v=z2IBSAuImyk

Video w. single operations multiplication and division only

Example 3.1.3

For each table below, guess the rule, that is, determine what is being done to the input x to get the output y? Then write a corresponding equation with x and y for each table.

a)

Input (x)	Output (y)
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

b)

Input (x)	Output (y)
-3	9
-2	6
-1	3
0	0
1	-3
2	-6
3	-9

c)

Input (x)	Output (y)
-3	-7
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

d) Now make up your own "guess the rule" problem. Then swap with a classmate to find an equation for each other's rule(s).

Formal Definition of a Function

A **function** is a relation between two sets, the **domain** (set/list of all input values) and **range** (set/list of all output values), such that each input value (in the domain) is matched to EXACTLY ONE output value (in the range).

If the input variable is x and the output variable is y, then we say “**y is a function of x**”.

Formally, we call the input variable the **independent variable** and the output variable the **dependent variable** (discuss why this makes sense).

Functions are powerful models for many real-life phenomena. If we can look at something that naturally has a relationship in which there is only one y for each x, then we might be able to capture that functional relationship algebraically (by this we mean with a formula) and then we can study the phenomenon by studying the algebraic model for the relationship.

Illustration

If we think of input (x) as a vending machine code and output (y) as the item that we get out of the machine then we consider this a function since each code should give us EXACTLY ONE item. Discuss this in terms of the image shown.

Image from https://brainly.com/textbook-solutions/q-27-analyzing-relationships-select-items-vending-machine-7#q-27-analyzing-relationships-select-items-vending-machine-5

What is the domain of this (vending machine) function? Record your answer below.

What is the range of this (vending machine) function? Record your answer below.

Can you think of how a different vending machine may NOT be a function? That is, how one code might give you two different items? Discuss.
The vending machine example provides a good introduction to the idea of a function. Now we will address examples with numbers as the input and output.

A good place to start with determining whether a (numerical) relationship is a function or not is with so called “arrow” or “mapping” diagrams that give a more visual representation.

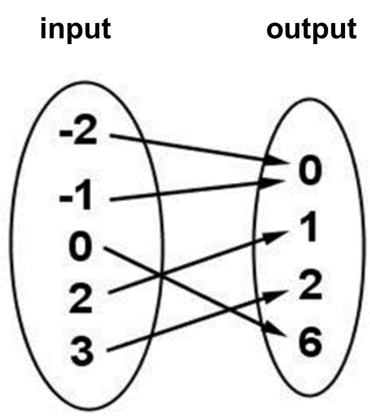
The following video addresses such diagrams and domain and range of the corresponding functions: https://www.youtube.com/watch?v=Io85GYd6-Yw

Watch this “function or not” video for examples with other representations (graph, table, points and mapping diagrams): https://www.youtube.com/watch?v=xSQFPbhT4Yc

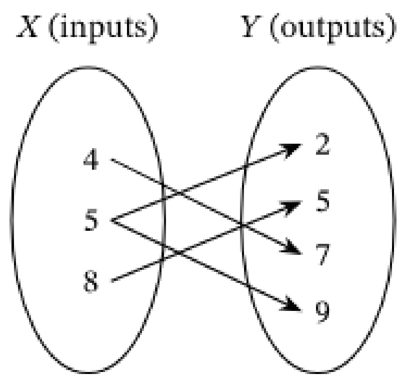
Example 3.1.4

Determine whether the following relationships represent functions or not and justify your answer. Also state the domain and range of each.

a)



b)



c)

x (input)	-3	-2	-1	0	1	2	3
y (output)	9	4	1	0	1	4	9

d)

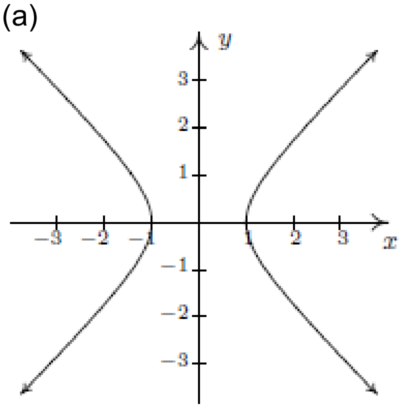
x (input)	3	2	1	0	1	2	3
y (output)	-9	-4	-1	0	1	4	9

Example 3.1.5

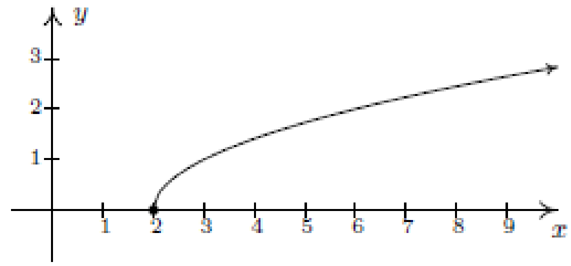
Use the Vertical Line Test to determine whether the following graphs express y as a function of x or not and justify your answer. Also state the domain and range of each.

Images from p.50 in OER http://www.lulu.com/shop/carl-stitz-and-jeff-zeager/functions-fitchburg-state-2017/paperback/product-23311630.html

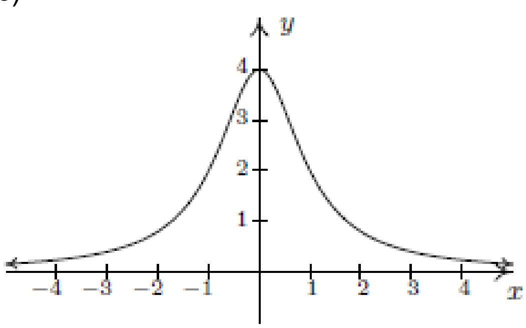
(a)



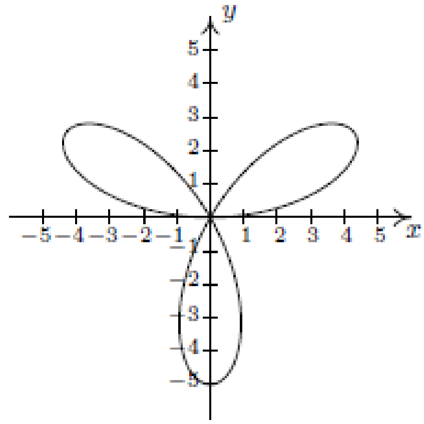
(b)



(c)



(d)



e) State the Vertical Line Test in your own words and explain why it works.

Summary: If each input value matches to EXACTLY ONE output value then the relationship IS a function. If there is an input value that matches to MORE THAN ONE output value then the relationship is NOT a function.

So far each representation has its own "test" for determining whether it is a function or NOT.

Mapping/Arrow diagrams: NOT a function if one input points to two or more outputs

Tables: NOT a function if any x-values repeat (assuming distinct y-values)

Graphs: NOT a function if a vertical line intersects the graph more than once

When it comes to equations….

If we can solve for y and get a single expression in terms of x ONLY then it IS a function.

If we cannot solve for y or if we try to solve for y and get more than one expression, then it is NOT a function.

Watch the following video for an example of an equation that does NOT represent y as a function of x https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-linear-equations-functions/cc-8th-function-intro/v/recognizing-functions-example-4

Example 3.1.6

Determine whether the following equations express y as a function of x or not and justify your answer.

a) $\lvert x\rvert + y = 5$	b) $x^2 - 4y^2 = 1$
c) $y = \sqrt{x-2}$	d) $x = \lvert y-2\rvert$

Finally we look at relationships expressed **in words** and determine if they are functions or not.

The vending machine illustration showed us such an example of a functional relationship between the input code and the output product.

Watch the following video for an example of a relationship that is NOT a function:
https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-linear-equations-functions/cc-8th-function-intro/v/recognizing-functions-example-5

Example 3.1.7

a) A teacher has his 25 students call off numbers 1, 2, 3, 4, 5 to put his class into 5 groups. Is the relationship between the student (input) and their group number (output) a function or not? Explain your answer.

b) A teacher has her students build as many rectangles as they can with 12 square tiles. Is the relationship between the area (input) of these rectangles and the perimeter (output) of these rectangles a function or not? Explain your answer.

Example 3.1.8

a) Describe a relationship in words that is a function. Identify your input and output.

b) Describe a relationship in words that is NOT a function. Identify your input and output.

3.1 Exercises

1) Assuming input is x and output is y, determine if the following relationships are functions or not and explain your answer. Also state the domain and range of each.

a)

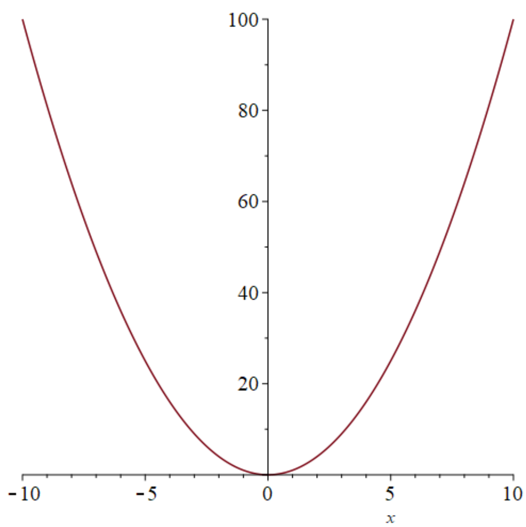
x	0	1	2	3	4	5	6
y	1	0	3	3	5	-5	7

b)

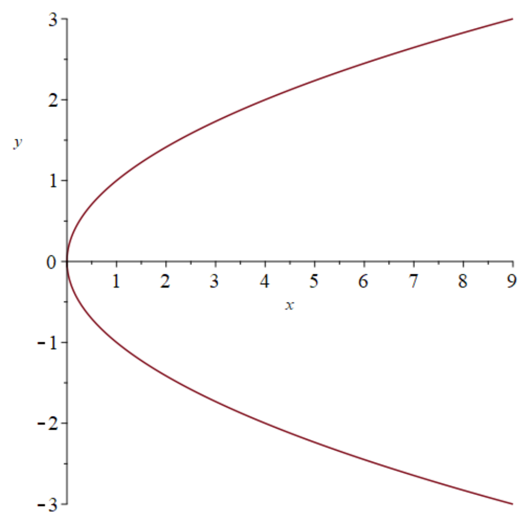
x	1	-1	0	-1	5	-5	2
y	1	1	2	3	4	4	8

2) Assuming input is x and output is y, determine if the following relationships are functions or not and explain your answer. Also state the domain and range of each.

a)



b)



3) Assuming input is x and output is y, determine if the following relationships are functions or not and explain your answer.

a) $x^2 + y^2 = 1$

b) $y = x^3$

4) Give one reason why the relationship between a domain of the set of your classmates and a range of the set of their favorite colors might not be a function.

5) Describe a real-world relationship that is a function and explain why. Also state what your input and output is and what the domain and range is.

6) Work through problems at the following links in Kahn Academy:

Recognizing Functions from Graphs

Recognizing Functions from Tables

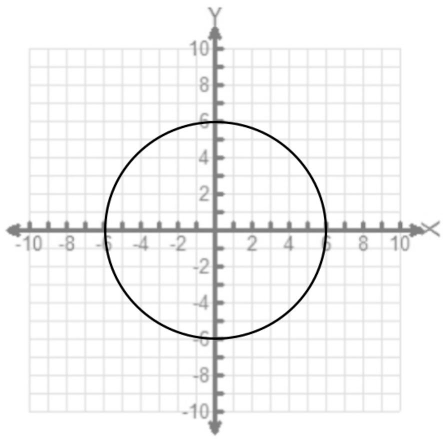
Watch the following video in Kahn Academy:

Recognizing Functions from Verbal description word problems

3.1 Extra Practice

For exercises 1-6, decide whether each graph is the graph of a function. Then determine domains and range.

1.

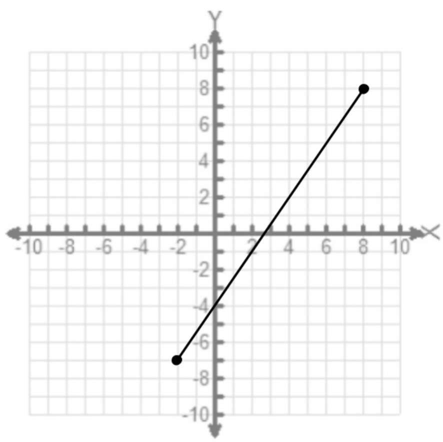


(a) Is it a function?

(b) Range:

(c) Domain:

2.

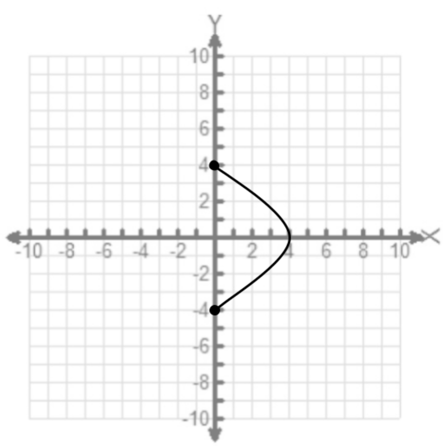


(a) Is it a function?

(b) Range:

(c) Domain:

3.

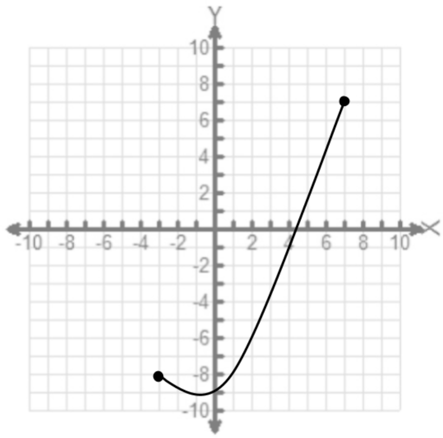


(a) Is it a function?

(b) Range:

(c) Domain:

4.

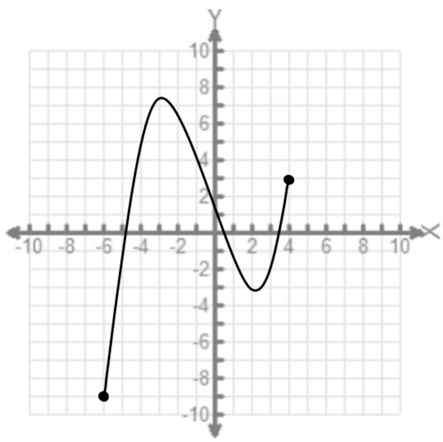


(a) Is it a function?

(b) Range:

(c) Domain:

5.

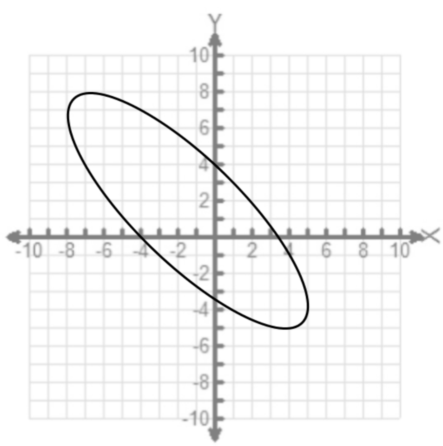


(a) Is it a function?

(b) Range:

(c) Domain:

6.



(a) Is it a function?

(b) Range:

(c) Domain:

For exercises 7-12, use each table to determine whether the relation is a function. Then determine the domain and range.

7.

x	2	4	6	8	10
y	1	3	5	7	9

(a) Is it a function?

(b) Range:

(c) Domain:

8.

x	2	2	4	4	6
y	-5	0	5	10	15

(a) Is it a function?

(b) Range:

(c) Domain:

9.

x	1	2	3	4	5
y	-5	-5	5	5	15

(a) Is it a function?

(b) Range:

(c) Domain:

10.

x	0	5	10	15	20
y	3	6	9	12	15

(a) Is it a function?

(b) Range:

(c) Domain:

11.

x	-9	7	0	3	-4
y	-2	12	-6	7	12

(a) Is it a function?

(b) Range:

(c) Domain:

12.

x	-4	9	2	-4	6
y	-10	8	2	-3	14

(a) Is it a function?

(b) Range:

(c) Domain:

For exercises 13-18, determine whether each relation is a function. Then determine the domain and range.

13. $\{(\mathbf{0}, -\mathbf{7}), (\mathbf{2}, \mathbf{5}), (-\mathbf{3}, \mathbf{1}), (-\mathbf{8}, \mathbf{0})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

14. $\{(\mathbf{3}, -\mathbf{5}), (\mathbf{8}, -\mathbf{6}), (\mathbf{3}, \mathbf{7}), (\mathbf{5}, \mathbf{9})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

15. $\{(-\mathbf{4}, \mathbf{7}), (\mathbf{2}, -\mathbf{3}), (\mathbf{7}, \mathbf{7}), (-\mathbf{5}, \mathbf{1})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

16. $\{(\mathbf{0}, -\mathbf{4}), (\mathbf{0}, \mathbf{2}), (\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{0})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

17. $\{(\mathbf{8}, -\mathbf{3}), (\mathbf{2}, -\mathbf{3}), (\mathbf{9}, -\mathbf{3}), (-\mathbf{1}, -\mathbf{3})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

18. $\{(\mathbf{9}, \mathbf{7}), (\mathbf{4}, \mathbf{3}), (-\mathbf{2}, \mathbf{0}), (\mathbf{9}, \mathbf{1})\}$

(a) Is it a function?

(b) Range:

(c) Domain:

Extra Practice with Functions as Equations

Determine if the following is an equation of a function or not. Explain your reasoning clearly.

1. $x + y^2 = 4$

2. $x^2 + y^2 = 4$

3. $|x - 1| + y = 2$

4. $y = x^2 - 1$

5. $x = \sqrt{y - 1}$

6. $|x| + |y| = 5$

3.2 Average Rate of Change

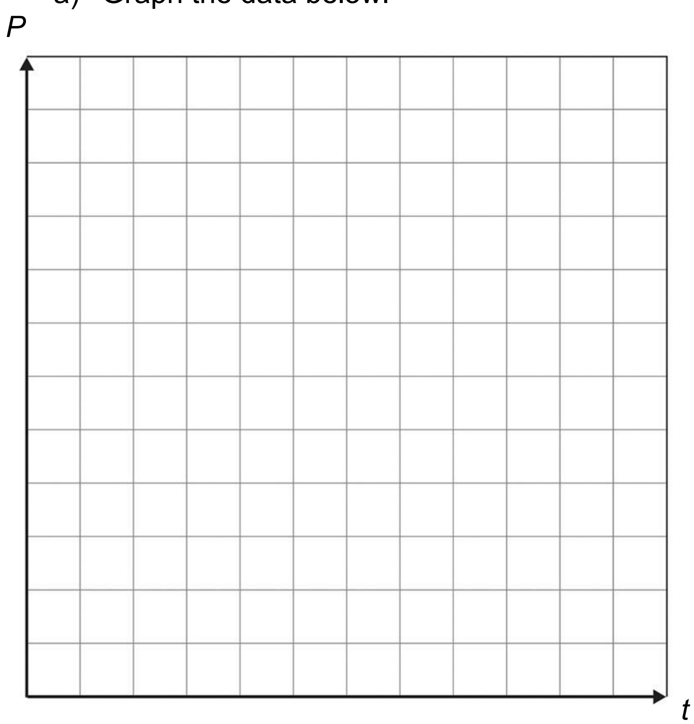
In this section we look at how the output changes as we change the input of a function. This is something we have already observed in many examples (e.g. how the figure grows as the figure number increases). Now we formalize the concept. Let us first look at a specific example.

Example 3.2.1

Consider the table below that gives the number of push-ups, *P*, Kenya has completed as a function of time *t* in seconds.

Time in seconds (input *t*)	0	10	20	30	40	50
Total number of push ups (output *P*)	0	5	25	50	64	70

a) Graph the data below.



b) By how much is the number of push-ups increasing per second on average from 10 seconds to 50 seconds? We figure this out by taking the total change in the number of push-ups during this time period and dividing by the total change in time. Show your work below.

c) What does the answer to part (b) tell us about the number of push-ups Kenya did? Explain in your own words.

We call the answer to Example 3.2.1 (b) the *average rate of change* of the number of push-ups from 10 to 50 seconds.

In general, the **average rate of change** of an output quantity on a given input interval is the change in the output divided by the change in the input on that interval. That is, the change in the dependent (output) variable divided by the change in the independent (input) variable.

This means the amount the output changes (on average) for each unit increase in the input (e.g. number of push ups per second)

In Example 3.2.1 we looked at $\dfrac{change\_in\_P}{change\_in\_t}$.

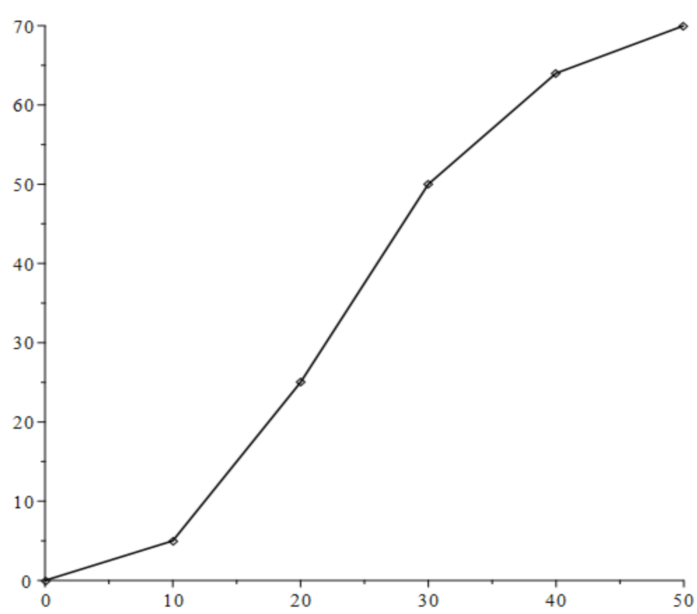
This formula should look somewhat familiar as the formula for SLOPE.

In particular, the slope of the line through the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$.

So the average rate of change of the number of push-ups from 10 to 50 seconds is equal to the slope of the line through the points (10, 5) and (50, 70).

Below is a sketch of the graph of P as a function of t with points connected by line segments.

Draw the line through points (10, 5) and (50, 70) and specify the slope of this line based on your answer in Example 3.2.1



Example 3.2.2

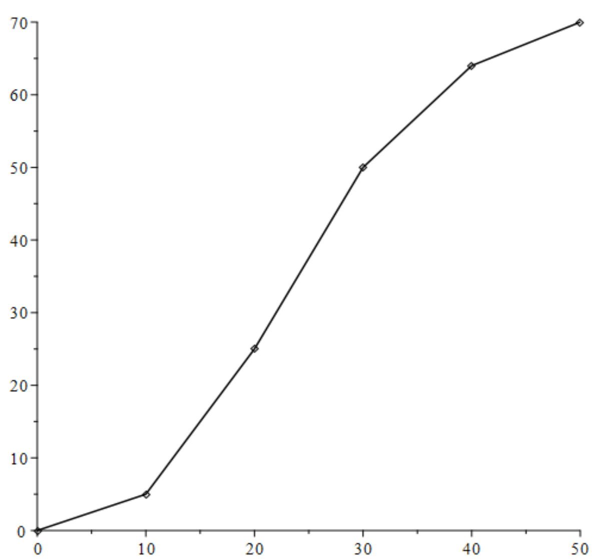
Reconsider the push up data copied below.

Time in seconds (input t)	0	10	20	30	40	50
Total number of push ups (output P)	0	5	25	50	64	70

a) What is the average rate of change of the total number of push-ups from 0 to 30 seconds?

b) What does the answer to part (a) tell us about the number of push-ups Kenya did?

c) Show (and explain) on the graph below what your answer to (a) means in terms of slope.



Function Notation: Read the "Function Notation" section of 1.2 Functions and work through the corresponding "checkpoint" exercises 1.51 - 1.58.

Below we use function notation to define average rate of change (that is, slope)

The **average rate of change** of a function f(x) from x = a to x = b is

$$\frac{f(b) - f(a)}{b - a}$$

Observe that this is the **slope** of the line through the points (a, f(a)) and (b, f(b)).

This is the same average rate of change formula as before but stated using function notation.

Example 3.2.3

Find the average rate of change of $f(x) = 3x^2 + 2x - 7$ from x = -4 to x = 2.

Example 3.2.4

Let $C(x) = x^2 - 10x + 27$ represent the cost, in hundreds of dollars, to produce *x* thousand pens. Find the average rate of change of cost as production increases from making 3000 to 5000 pens. Then explain what the answer tells us about pen production.

Supplemental material

Supplemental Material Section 1.4: Slope and Rate of Change

Connecting Functions to Sequences: Section 9.1: Sequences . Read and work through the corresponding "checkpoint" exercises.

3.2 Exercises

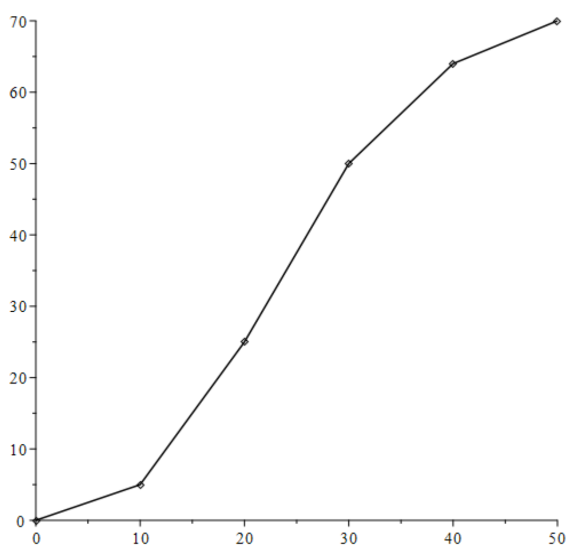
1) Reconsider the push up data copied below.

Time in seconds (input *t*)	0	10	20	30	40	50
Total number of push ups (output *P*)	0	5	25	50	64	70

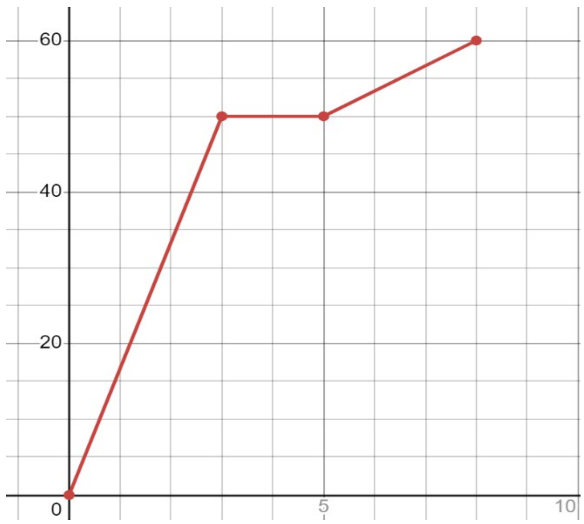
a) What is the average rate of change of the total number of push-ups from 20 to 40 seconds?

b) What does the answer to part (a) tell us about the number of push-ups?

c) Show (and explain) on the graph below what your answer to (a) means in terms of slope.



2) Consider the graph below representing the number of customers served at a Chick-fil-A as a function of time in hours.



a) Find the average rate of change from 0 to 3 and explain what information this gives you about the customers.

b) Find the average rate of change from 0 to 7.5 and explain what information this gives you about the customers.

c) What do you think the horizontal line segment signifies?

3) The height *H,* of an object dropped from the roof of an eight story building is given by the following equation $H = -16t^2 + 64$ where *H* is the height of the object in feet, t seconds after it is dropped. Find the average rate of change of *H* from 0 to 2 seconds and explain what it tells us about the height of the object.

4) The temperature F in degrees Fahrenheit t hours after 6 AM is given by the following equation $F(t) = -\frac{1}{2}t^2 + 8t + 32.$

   a) Find the average rate of change of F from t = 4 to t = 8 and explain what it tells us about the temperature.

   b) Find the average rate of change of F from t = 8 to t = 12 and explain what it tells us about the temperature.

   c) Find the average rate of change of the temperature between 10 AM and 6 PM.

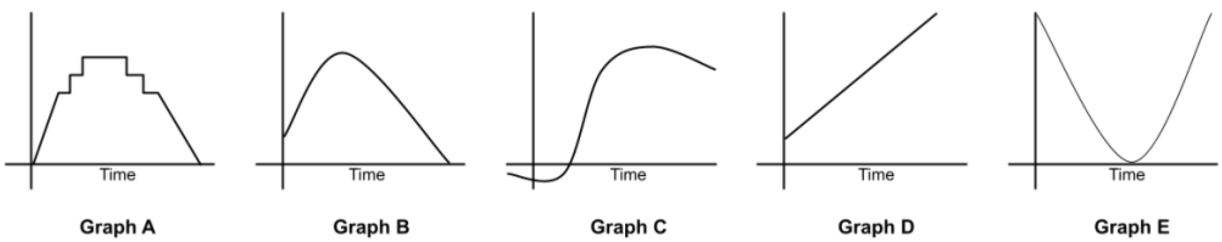
3.3 Reading Function Graphs

In this section we determine graphs that correspond to stories and stories that correspond to graphs. The focus is not as much on the points and numbers on the graph but rather the general shape of the graph, how it changes and what this means in terms of the story scenarios. This is a skill that is very useful for elementary school students to get a broader understanding of functional relationships.

Let us start with the lesson below from *Everyday Mathematics* for the fifth grade.

Example 3.3.1

Consider the graphs below



Match each scenario below with a graph above and be ready to explain your answers.

1. A burrito is removed from the freezer. It is cooked in a microwave oven. Then it is placed on the table.

   Which graph shows the temperature of the dinner at different times? Graph:____________________

2. Sarya runs water into his bathtub. He steps into the tub, sits down, and bathes. He gets out of the tub and drains the water.

   Which graph shows the height of water in the tub at different times? Graph:____________________

3. A water balloon is thrown straight up in the air.

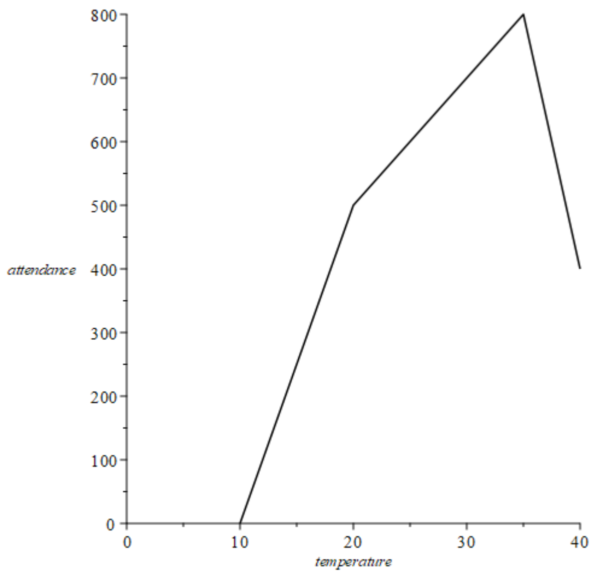
   a. Which graph shows the height of the water balloon from the time it is thrown until the time it hits the ground? Graph:____________________

   b. Which graph shows the speed of the water balloon at different times? Graph:____________________

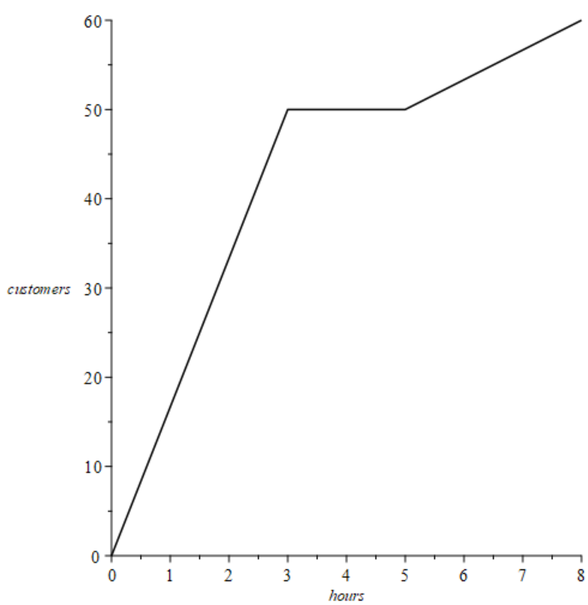
Example 3.3.2

Write a story to go along with each of the following graphs. In your stories, address changes in slope, where the graph intersects the axes, high and low points and where the graph is going up, going down and staying flat.

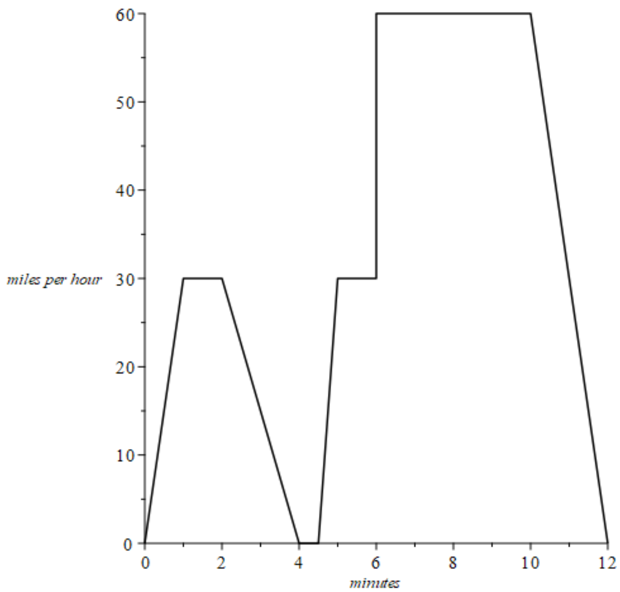
a) Attendance at a beach in Saudi Arabia



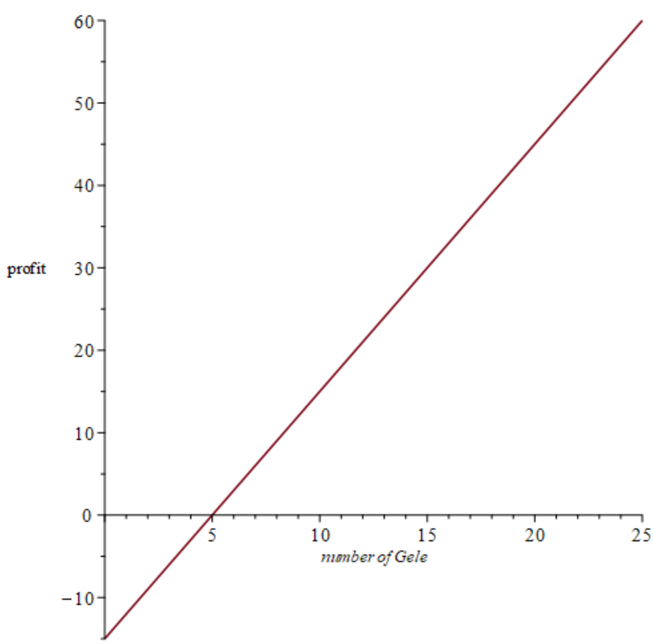
b) Customers served at an Indian restaurant



c) Driving home from school



d) Profit earned from the sale of Gele (African head wraps)



Example 3.3.3

For the sketches below, you do not have to put numbers on your graphs.

a) Halfway from your English class to your math class, you realize that you left your math book in the classroom. You retrieve the book, then walk to your math class. Graph the distance between you and your English classroom as a function of time, from the moment you originally leave the English classroom until you reach the math classroom.

b) Toni drives from home to meet her friend at the gym, which is halfway between their homes. They work out together at the gym; then they both go to the friend's home for a snack. Finally, Toni drives home. Graph the distance between Toni and her home as a function of time, from the moment she leaves home until she returns.

c) Forrest leaves his house to go to school. For each of the following situations, sketch a possible graph of Forrest's distance from home as a function of time.

(i) Forrest walks at a constant speed until he reaches the bus stop.

(ii) Forrest walks at a constant speed until he reaches the bus stop; then he waits until the bus arrives.

(iii) Forrest walks at a constant speed until he reaches the bus stop, waits until the bus arrives, then the bus drives him to school at a constant speed.

Concavity

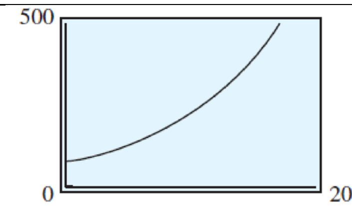
Illustration

Consider the two functions described below in part (a) and (b). Both functions are increasing, but in different ways.

a) The number of flu cases reported at a city medical center during an epidemic is an increasing function of time, and it is growing at a faster and faster rate.

x	0	2	5	10	15
y	70	89	123	217	383

The number of flu cases is increasing more and more as time goes on. We can see this by computing the rate of change over successive time intervals as shown below

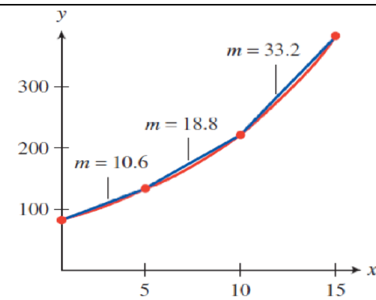


(A)

$x = 0$ to $x = 5$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{123 - 70}{5 - 0} = 10.6$

$x = 5$ to $x = 10$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{217 - 123}{10 - 5} = 18.8$

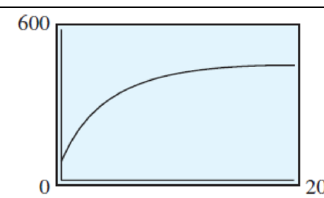
$x = 10$ to $x = 15$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{383 - 217}{15 - 10} = 33.2$



b) The temperature of a potato placed in a hot oven increases rapidly at first, then more slowly as it approaches the temperature of the oven.

x	0	2	5	10	15
y	70	219	341	419	441

The temperature of the potato is increasing less and less as time goes on. We can see this by computing the rate of change over successive time intervals as shown below

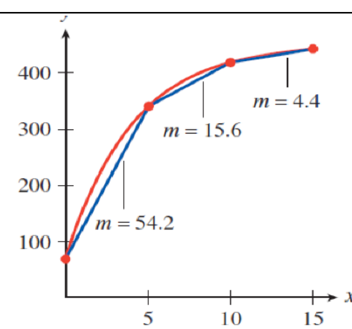


(B)

$x = 0$ to $x = 5$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{341 - 70}{5 - 0} = 54.2$

$x = 5$ to $x = 10$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{419 - 341}{10 - 5} = 15.6$

$x = 10$ to $x = 15$: $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{441 - 419}{15 - 10} = 4.4$



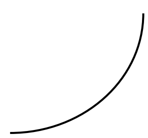
In the illustration we see that the slopes of the graph are changing as time goes on. This is why the graph is curved. In contrast, when we are looking at a straight line, the slope is always the same. Later chapters address linear and nonlinear functions more formally.

For now we focus on the rate of change implications of curved functions.

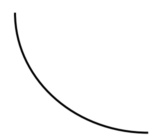
In general…

A graph (or portion of a graph) is **concave up** if it curves upward like a bowl or half bowl shape. This means that the function is increasing at an increasing rate (i.e. increasing faster and faster) or decreasing at a decreasing rate (i.e. decreasing slower and slower). Both situations are illustrated below.

Increasing at an increasing rate

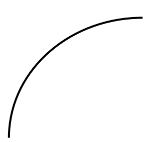


Decreasing at a decreasing rate

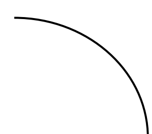


A graph (or portion of a graph) is **concave down** if it curves downward like an upside down bowl or half bowl shape. This means that the function is increasing at a decreasing rate (i.e. increasing slower and slower) or decreasing at an increasing rate (i.e. decreasing faster and faster). Both situations are illustrated below.

Increasing at a decreasing rate



Decreasing at an increasing rate



The implications of these types of curves should be addressed when reading graphs or writing stories for graphs with such curves.

Example 3.3.4

Sketch a possible graph to illustrate the height above the ground of a rubber ball dropped from the top of a 10-foot ladder, as a function of time. Show at least 4 bounces. Label your axes appropriately.

Example 3.3.5

David turns on the oven and it heats up steadily until the proper baking temperature is reached. The oven maintains that temperature during the time David bakes a pot roast. When he turns the oven off, David leaves the oven door open for a few minutes, and the temperature drops fairly rapidly during that time. After David closes the door, the temperature continues to drop, but at a much slower rate. Graph the temperature of the oven as a function of time, from the moment David first turns on the oven until shortly after David closes the door when the oven is cooling.

3.3 Exercises

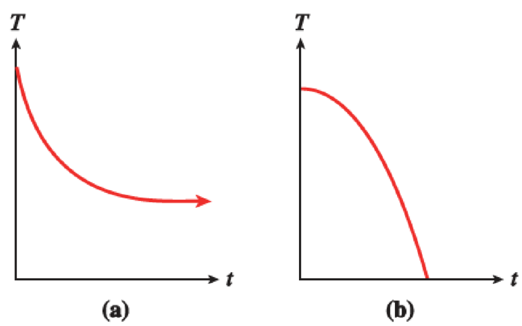
1) Erin walks from her home to a convenience store, where she buys some cat food, and then walks back home. Sketch a possible graph of her distance from home as a function of time.

2) While bicycling from home to school, Greg gets a flat tire. He repairs the tire in just a few minutes but decides to backtrack a few miles to a service station, where he cleans up. Finally, he bicycles the rest of the way to school. Graph the distance between Greg and his home as a function of time, from the moment he leaves home until he arrives at school.

3) Sketch a possible graph of the height of your head above the ground during a ride on a Ferris wheel as a function of time. Show at least two full rotations.

4) After you leave your math class, you start off toward your music class. Halfway there you meet an old friend, so you stop and chat for a while. Then, you continue to the music class. Graph the distance between you and your math classroom as a function of time, from the moment you leave the math classroom until you reach the music classroom.

5) Francine bought a cup of cocoa at the cafeteria. The cocoa cooled off rapidly at first, and then gradually approached room temperature. Which graph below, (a) or (b), more accurately reflects the temperature of the cocoa as a function of time? Explain.



Select statements below that justify your answer by circling all that apply.

a) The graph has a steep negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa.

b) The graph becomes closer to a horizontal line, corresponding to the cocoa approaching room temperature.

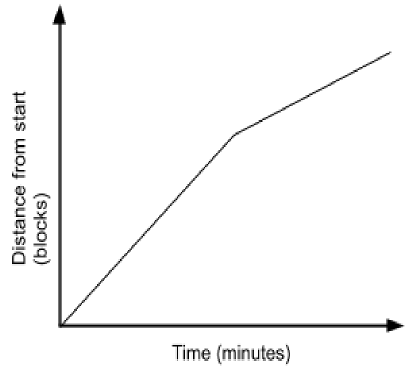
c) The graph has a slight negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa.

d) The graph becomes steeper and steeper, corresponding to the cocoa approaching room temperature.

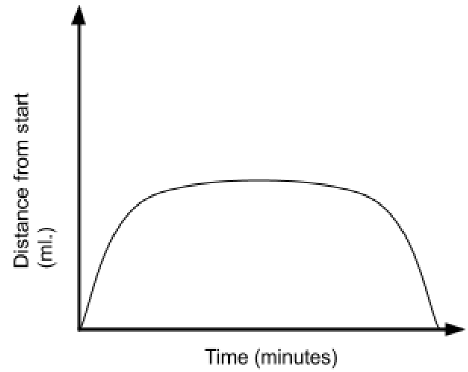
Is the graph you chose concave up or concave down?

6) Match each graph below with one of the scenarios that follow. Write 1-2 sentences for each scenario explaining why you chose the graph you did.

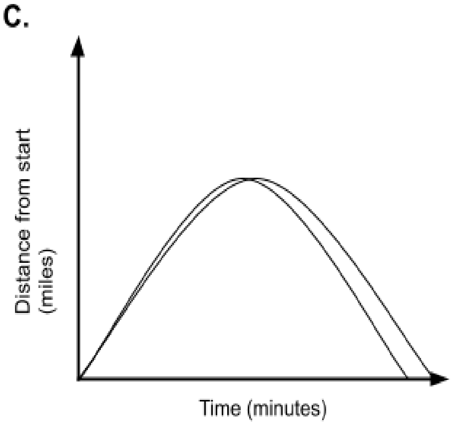
A.



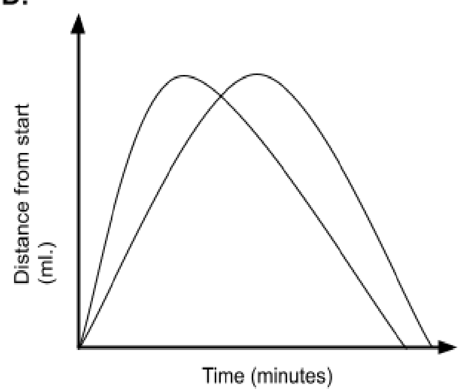
B.



C.



D.



- Kelly and Cindy sprinted from one end of the gym to the other end. They then jogged back to where they started. Cindy sprinted faster than Kelly.

- John walked to school. The second half of his walk was uphill, so he walked at a slower pace.

- Rachel drove to the dentist's office. Her appointment lasted for 45 minutes, and then she headed home.

- Mike and Ron rode their bikes from Middletown to Centerville and back. Soon after they began, Mike was always ahead of Ron.

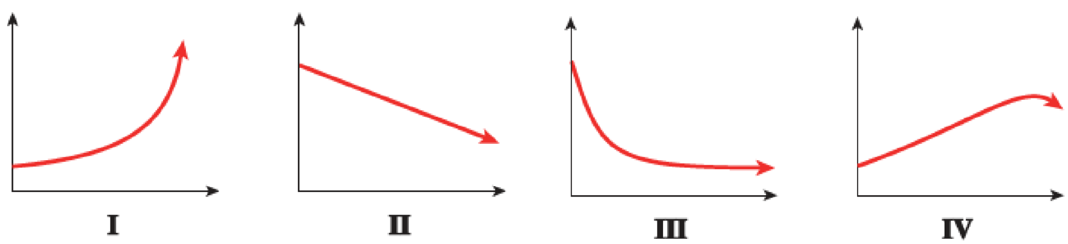
7) Match each situation below with graph I, II, III or IV.

a) Unemployment was falling but is now steady.

b) Inflation, which rose slowly until last month, is now rising rapidly.

c) The birthrate rose steadily until 1990 but is now beginning to fall.

d) The price of gasoline has fallen steadily over the past few months.



Chapter 3 Wrap Up

In this chapter we addressed functional relationships in general, the rate of change of functions and the implications of this in real life scenarios. These concepts are ubiquitous in all aspects of life and fundamental to mathematical modeling. Let us not forget that this all started with algebraic thinking and patterns! In subsequent chapters we address some very specific types of functions, beginning with linear functions in chapter 4.