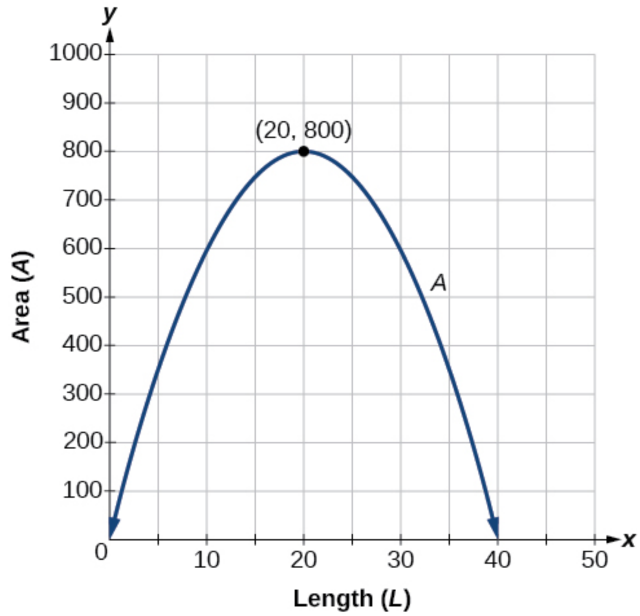
Chapter 5 QUADRATIC FUNCTIONS



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In this chapter…

5.1 Quadratic Expressions and Area
5.2 Quadratic Functions and Models

We previously studied linear models and several applications. However, many real world relationships are not linear. In this chapter we look at quadratic functions which are a common type of function that is non-linear, that is, the slope/rate of change is NOT constant. Such functions are used to model many relationships as we will see.

Although quadratic functions are not formally studied in elementary school mathematics, students may explore patterns that involve quadratic relationships.

Check out the lesson below from Everyday Mathematics (University of Chicago School Mathematics Project, 1995) in which second graders build square arrays that represent numbers multiplied by themselves and then explore the numerical patterns.

Lesson from Everyday Mathematics

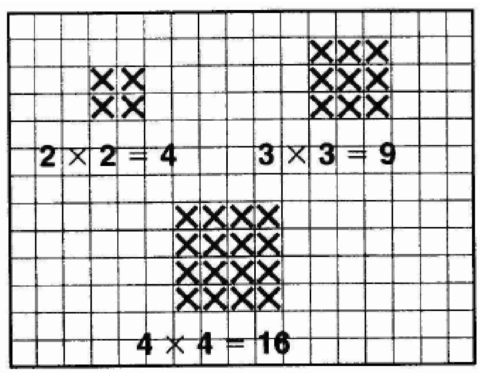
Work in a small group

Materials:

Centimeter grid paper
Centimeter cubes or pennies
Tape

Directions:

1. *Each person chooses a different number from 2 to 10*
2. *Build an array that shows your number multiplied by itself. Use pennies or centimeter cubes.*
3. *Draw each array on centimeter grid paper. Write a number model under each array.*



4. *Make and record and a few more arrays. On a blank sheet of paper, make a table like the one below. Begin with the smallest factors. Record them in order: 2 x 2, 3 x 3, 4 x 4 and so on.*

Array (factors)	**Total (products)**
2×2	4
3×3	9
4×4	16

5. *Continue working together. Build arrays with cubes or pennies for larger and larger numbers. Draw the arrays on grid paper. You may need to tape pieces of grid paper together for the larger arrays.*
6. *Record the factors and products for the larger numbers in your table. Look for number patterns as studied in earlier chapters.*

Such activities expose early elementary students to nonlinear relationships and also link to growing patterns as studied in earlier chapters.

In middle school, students explore other quadratic relationships such as the relationship between the radius and the area of a circle or the relationship between the base of a rectangle and its area which we will explore later in this chapter.

As a reminder, it is important for elementary teachers to have an in-depth understanding of elementary AND middle school mathematics. In particular, the Massachusetts Department of Elementary and Secondary Education (DESE) requires that teacher candidates for K-6 have mastery of concepts in grade K - 8. See https://www.doe.mass.edu/edprep/domains/instruction/smk-guidelines.docx

5.1 Quadratic Expressions & Area

Pre-Class Work

1) Watch the following videos on the distributive property. Find something about the use of symbols in the second video that may be confusing to viewers. Explain below.

   Distributive property in arithmetic: https://www.youtube.com/watch?v=VZ0jG3W53nE

   Distributive property in algebra: https://www.youtube.com/watch?v=v-6MShC82ow

2) Watch this video demonstrating how area may be used to multiply two digit and three digit numbers.

3) Watch the following video on the FOIL method: The FOIL Method. Find a point in the video where the person talking uses the wrong terminology/word. Explain below.

4) Read and work through checkpoint exercises at Section A.8: Factoring Quadratic Trinomials as needed for review on factoring. Be prepared to ask questions in class.

The association between arithmetic and geometry is ancient. A number can be thought of as a length, and addition can be thought of as combining lengths.

Areas of rectangles can be found by multiplying the side lengths. Of special interest are squares (rectangles with equal side lengths). The area of a square can be found by multiplying those equal side lengths together. And that is why $x \cdot x$ is commonly referred to as “x squared”, originally meaning “take the length of x and make a square with that side length and take its area”. Equations involving only constant, x, and x^2 terms are called quadratic equations. The word quadratic refers to the square (“quad” is Latin for “four”, for the four-sided square).

In this section we focus on a geometric way of looking at quadratic expressions. We illustrate the product of two binomials or a monomial (one term) and a binomial (two terms) with area, using a technique similar to that shown in pre-class work #2 video.

In particular, our goal will be to build rectangles whose area represents these products. This will enable us (and our future students) to see why the FOIL method works and how the distributive property plays an important role.

To do this we will use three different types of rectangles shown below:

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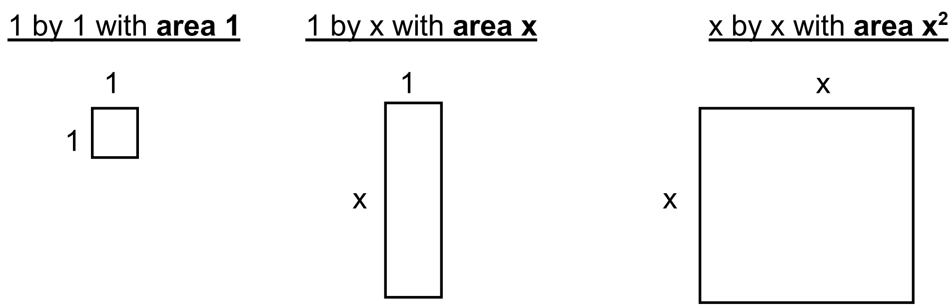
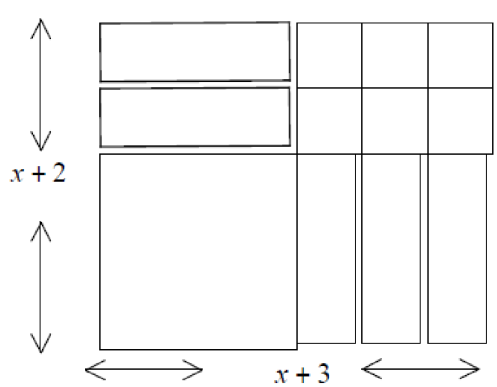


(Handout algebra tiles, if available)

Assume each corresponding algebra tile has the areas given above.

Example 5.1.1

Use your algebra tiles to form a rectangle with length (x + 3) and width (x + 2) as shown below. Label each tile with its area. Then add all the areas together and state your answer below.



Your total area should be equal to the length times the width, that is (x+2)(x+3).

Now use the FOIL method to confirm your answer. Show your work below.

How is the FOIL method related to the distributive property? Looking at the example above, we can view it as follows:

Distribute (x+3) to x and 2: (x+2)(x+3) = x(x+3) + 2(x+3)

Distribute x into (x+3) and 2 into (x+3): $x^2 + 3x + 2x + 6$

Add like terms: $x^2 + 5x + 6$

So, FOIL is really applying the distributive property twice.

Now you try.

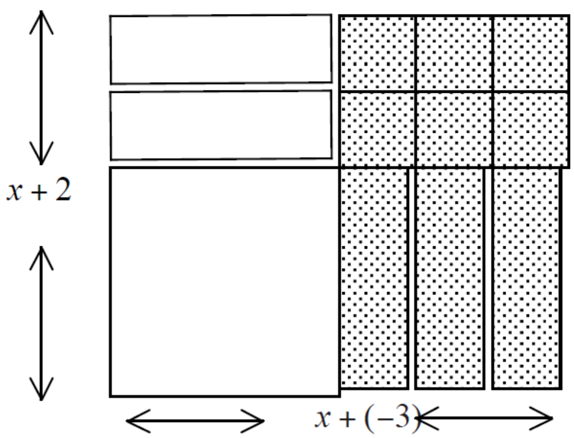
Multiply (x+2)(x+3) in a similar way by first distributing the (x + 2) to x and 3. Show work below.

How does the algebra tile method relate to the distributive property? Discuss with a partner. Use Example 5.1.1 to explain and make notes below**.**

When multiplying binomials with a negative term, we can **use shading** as shown in Example 5.1.2 **to distinguish area representing a negative product**.

Example 5.1.2

Consider the algebra tile representation of the product (x+2)(x-3) below. Observe that the shaded tiles correspond to multiplying a positive term by a negative term resulting in a negative product. Label each tile with its area and put a negative sign in front of areas corresponding to a shaded tile. Add all the areas together and state your answer below.



Now use the FOIL method to confirm your answer. Show your work below.

Example 5.1.3

Sketch algebra tile rectangles for each of the following products. Use algebra tiles as needed to help. Be sure to label the dimensions of your rectangles as we did in previous examples. Then, show the sum of the rectangle areas in expanded form before simplifying to get the final answer. Work with a partner - each pair will be assigned one part below to work on and share with class. If you finish your part early, try another part.

a) $(x+3)(x+4)$

b) $(-2x)(x + 4)$

c) $(x + 5)(x - 2)$

d) $(x - 2)(x - 1)$

e) $(2x + 3)(x + 1)$

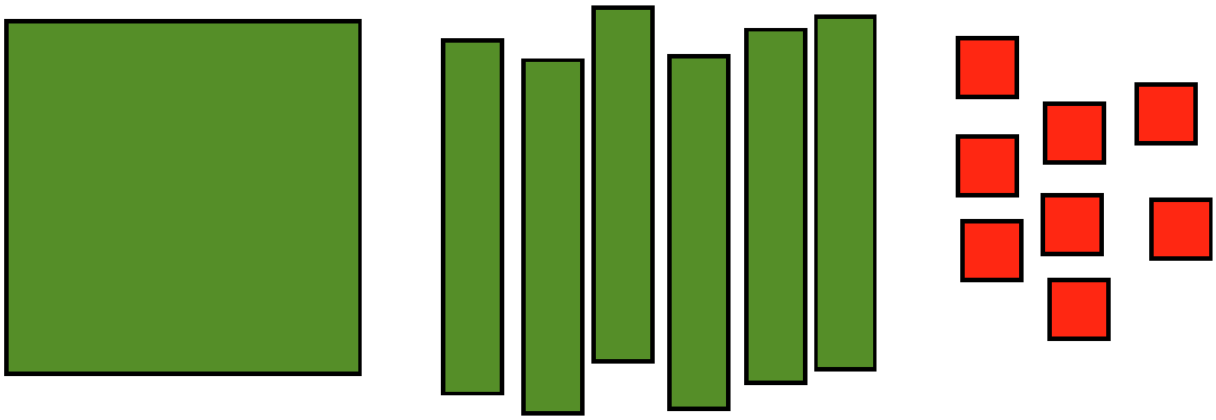
f) $(3x - 2)^2$

Now let us use tiles and rectangle areas to factor.

Watch the following video that explains how the process works:
https://www.youtube.com/watch?v=11UijrrceDg

Example 5.1.4

Consider the quadratic expression $x^2 + 6x + 8$. Observe that this expression may be represented as the collection of tiles below



Use these algebra tiles to try to build a rectangle. Sketch your rectangle below and label the side lengths. The side lengths of your rectangle will be the factors of $x^2 + 6x + 8$.

Example 5.1.5

Consider the quadratic expression $2x^2 - 3x + 1$. Use corresponding algebra tiles to build a rectangle and sketch your rectangle below. Make sure when you make your sketch you shade in the appropriate part to distinguish the negative term -3x. Use your rectangle to factor $2x^2 - 3x + 1$.

Example 5.1.6

Use algebra tiles to factor the following expressions as we did in previous examples. Sketch your rectangles below, label the side lengths, and state the corresponding factors of the quadratic. Work with a partner - each pair will be assigned one part below to work on and share with class. If you finish your part early, try another part.

a) $x^2 + 10x + 25$

b) $2x^2 + 7x + 6$

c) $2x^2 + 5x - 3$

d) $4x^2 + 9x + 2$

e) $6x^2 + 13x + 5$

5.1 Exercises

1) Use algebra tiles to model the following products. Sketch the corresponding rectangles and label them appropriately.
   a) $(2x - 1)(x + 4)$

   b) $(1 - 3x)(3 + x)$

   c) $(2 + x)(3 - x)$

   d) $(2x + 3)(3x - 1)$

2) Use algebra tiles to factor the following quadratics. Sketch and label the corresponding rectangles.

a) $x^2 - 5x + 6$

b) $3 + 2x - x^2$

c) $2x^2 + 13x + 6$

d) $2x^2 + x - 3$

5.2 Quadratic Functions & Models

Pre-Class Work

1) Watch this short video on linear vs quadratic equations and create some of your own examples to share in class.
2) Read and work through checkpoint exercises at the following links as needed for review on solving quadratic equations. This pre-class work will be needed later in the section when we address x-intercepts of quadratic functions.

   Subsection: Quadratic Formula

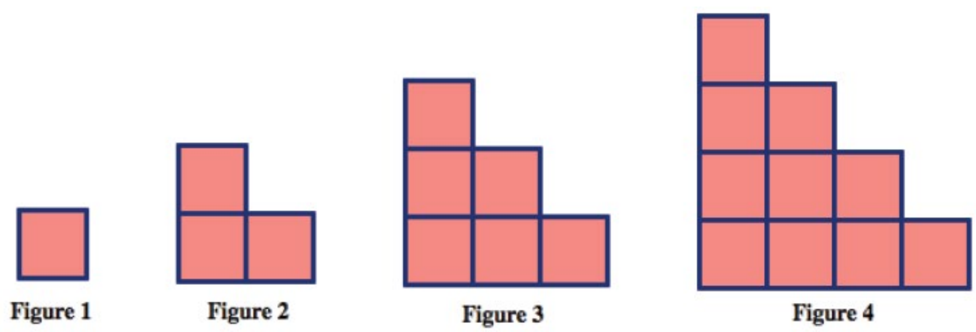
   Subsection: Solving Quadratic Equations by Factoring

   Note that although we will focus on quadratics that can be factored, not all quadratics can be factored which is one reason why the quadratic formula is so important.

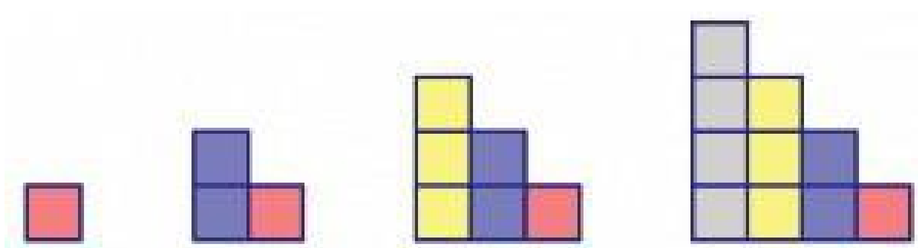
We introduce this section by revisiting a growing pattern (from chapter 2) that may be modeled by a quadratic relationship. Later in the section we will address quadratic functions more formally.

Example 5.2.1

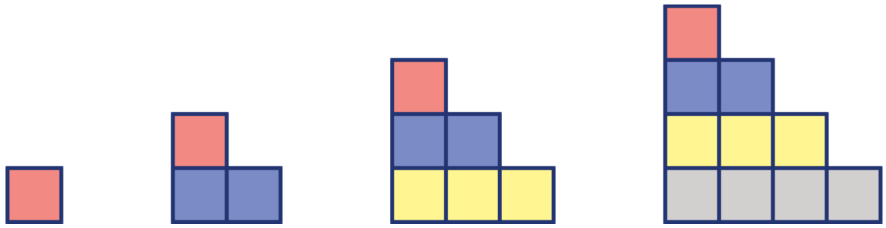
Consider the growing pattern below.



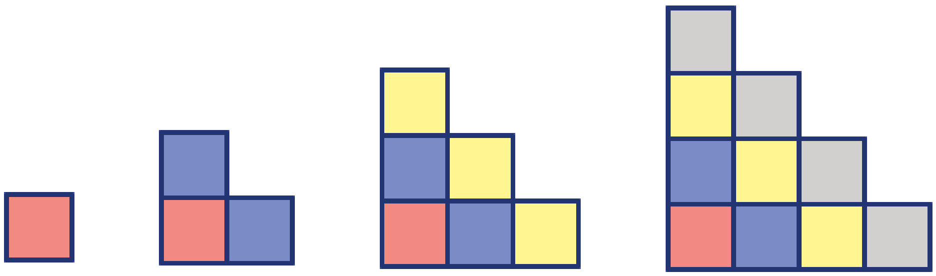
Below are some pictures that students drew to describe how the pattern was growing.



Ali's picture



Michael's picture



Kelli's picture

Use the students' pictures above (or your own method of seeing the growing pattern) to answer the following questions. Explain how you got your answers.

- How many tiles would you need to build the 5th figure in the pattern?

- How many tiles would you need to build the 10th figure in the pattern?

- How can you compute the number of tiles in any figure in the pattern?

Hy looked at the pattern in a different way by doubling each figure to build rectangles as follows:

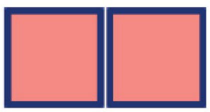


Figure 1

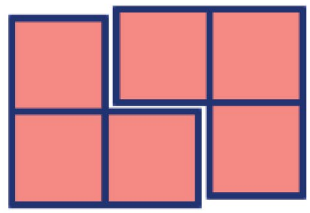


Figure 2

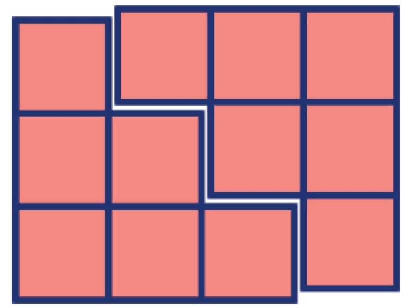


Figure 3

Hy's picture above

a) Draw similar pictures for figure 4 and figure 5 and use the dimensions of the rectangles to determine the number of tiles in these figures in the original pattern.

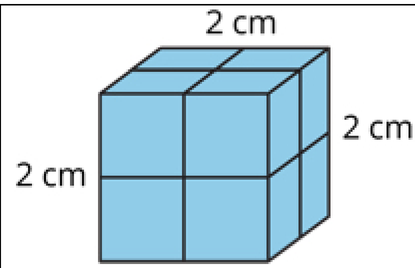
b) Use the dimensions of the corresponding rectangle to calculate the number of tiles needed to build the 10th figure and the 100^{th} figure in the pattern.

c) Write a formula that calculates the number of tiles in figure n. Sketch a corresponding figure below.

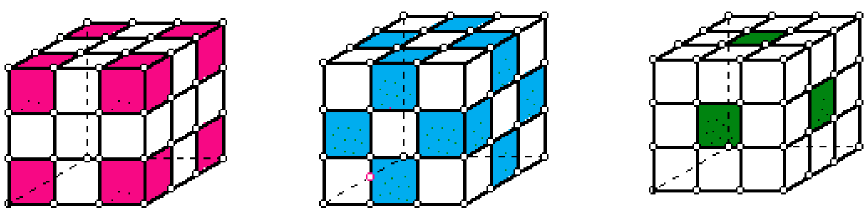
Discussion: How might you show/describe this pattern to a visually impaired student?

Example 5.2.2

Break up into pairs/groups - instructor handout snap blocks.

Consider a 2 by 2 by 2 cube (length = 2, height = 2 and width = 2) made up of 8 unit blocks (1 by 1 by 1) as shown to the right. Observe that if the cube is dipped in paint then each of the unit blocks has 3 of its faces painted. Build this with your snap cubes to understand.	

a) Build a 3 by 3 by 3 cube in your groups using snap blocks. If this 3 by 3 by 3 cube is dipped in paint, the corresponding unit blocks used to build it may have paint on 3, 2, 1 or 0 faces as shown in the picture below.



The cube on the left has shaded/colored the corner unit blocks which are each painted on 3 faces.

The cube in the center has shaded/colored the unit blocks between the corners, which are each painted on 2 faces.

The cube on the right has shaded/colored the unit blocks in the middle of each face, which are each painted on 1 face.

Determine the following…

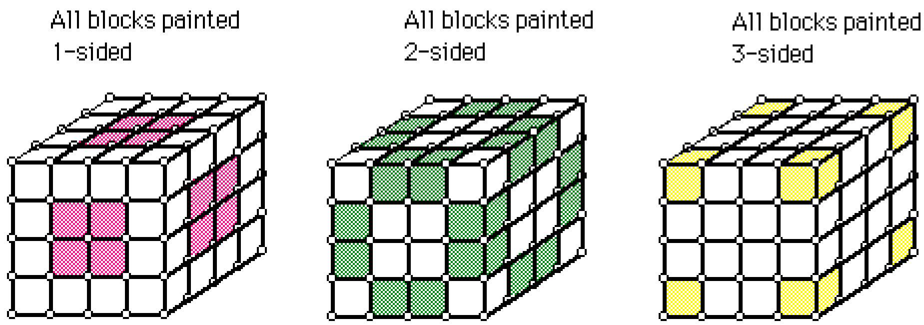
Number of unit blocks painted on 3 faces =

Number of unit blocks painted on 2 faces =

Number of unit blocks painted on 1 face =

Number of unit blocks painted on 0 faces =

b) Now build a 4 by 4 by 4 cube. If this 4 by 4 by 4 cube is dipped in paint, determine the following (using snap blocks and the picture below to help).



Number of unit blocks painted on 3 faces =

Number of unit blocks painted on 2 faces =

Number of unit blocks painted on 1 face =

Number of unit blocks painted on 0 faces =

c) Using the information above and, assuming similar scenarios with larger cubes, fill in the table below:

Dimension of cube	Number of unit blocks painted on 3 faces	Number of unit blocks painted on 2 faces	Number of unit blocks painted on 1 face	Number of unit blocks painted on 0 faces	Total number of unit blocks
2	8	0	0	0	8
3					
4					
5					
10					

n					

d) Considering your results in the last row of the table, name the type of functional relationship (linear, quadratic, other) between n and each of the following, and discuss why each answer makes sense.

Number of unit blocks painted on 3 faces

Number of unit blocks painted on 2 faces

Number of unit blocks painted on 1 face

Number of unit blocks painted on 0 faces

Now we formally define quadratic functions and explore some properties.

A **quadratic function** is one that may be expressed in the form $y = ax^2 + bx + c$ where a, b and c are real number constants and $a \neq 0$.

Recall linear function form $y = mx + b$.

What is the difference between a linear and a quadratic function (in terms of the equation)?

There are three main forms of quadratic functions:

1) General/expanded form: $y = ax^2 + bx + c$
2) Standard/vertex form: $y = a(x - h)^2 + k$
3) Factored form: $y = a(x - ?)(x - ?)$

Illustration

Quadratic functions in various forms (state the form next to each one)

a) $y = 2(x-1)^2 - 2$

b) $y = 2x^2 + 8x + 5$

c) $y = -2(x-3)(x+2)$

What do graphs of quadratic functions look like? Sketch below. What are they called?

Terminology

The **vertex** of a quadratic function is the lowest or highest point on the parabola depending on whether the graph opens up or down. *Mark vertex on sketches above.*

The **axis/line of symmetry** is the vertical line that goes through the vertex. The parabola is a mirror image of itself through this line. If the vertex is (h, k) then the axis of symmetry is x = h.

Recall…..

y-intercept: point where x = 0. To find it we can set x = 0 and solve for y, then the point (0,y) will be the y-intercept.

x-intercept: a point where y = 0. To find it we can set y = 0 and solve for x, then the point (x,0) will be an x-intercept.

Note that intercepts are points (ordered pairs), not single values!

Questions for discussion

Does every parabola have a y-intercept? Can there be more than one y-intercept? Explain your reasoning.

Does every parabola have x-intercepts? How many x-intercepts can a parabola have? Explain your reasoning.

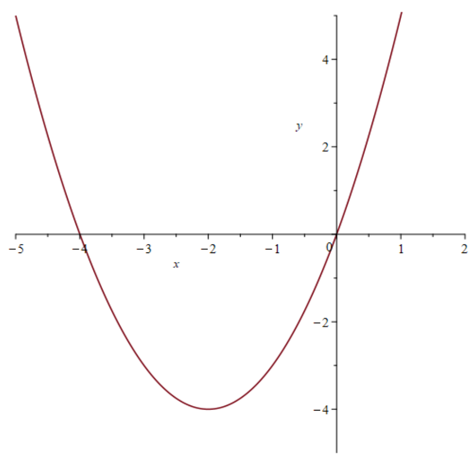
Example 5.2.3

The following exercise is a modified version of a problem designed for middle school students, from *Connected Mathematics, Frogs, Fleas and Painted Cubes: Quadratic Relationships* (Lappan, Fey, Fitzgerald, Friel & Phillips, 1998).

Match each graph below to one of following equations:

a) $y = x^2$	e) $y = x(x-4)$
b) $y = (x+2)(x-3)$	f) $y = x(x+4)$
c) $y = 2x(x+4)$	g) $y = (x+3)(x+2)$
d) $y = x(4-x)$	h) $y = (x+3)(x-3)$

Fill in the equation and other information for each graph below

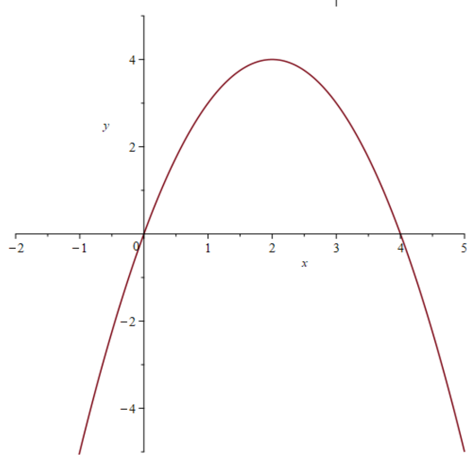


Equation:

x-intercepts:

line of symmetry:

max or min point (aka vertex):

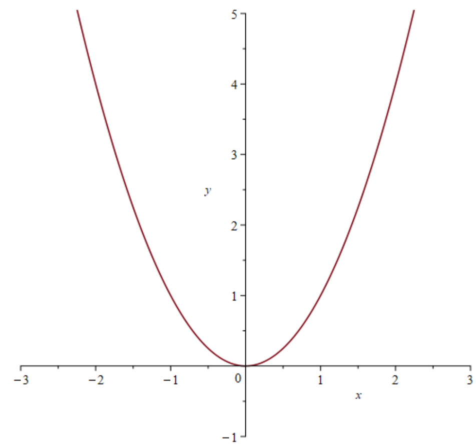


Equation:

x-intercepts:

line of symmetry:

max or min point (aka vertex):

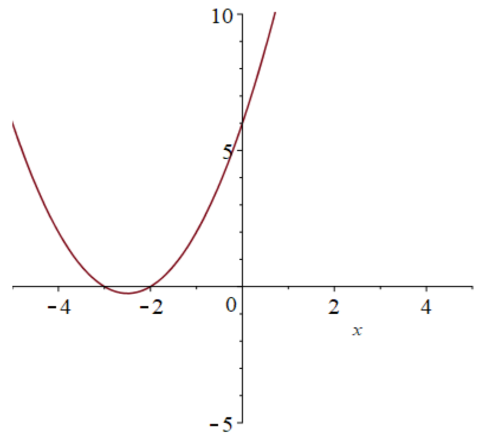


Equation:

x-intercepts:

line of symmetry:

max or min point (aka vertex):



Equation:

x-intercepts:

line of symmetry:

max or min point (aka vertex):

y-intercept:

Explain (in general) how you can determine the x-intercepts from the equation.

What other features of the graph can be determined by the equation? Explain.

In general, if (r, 0) is an x-intercept of a quadratic function then (x – r) is a factor of the quadratic.

In summary, if r_1 and r_2 are x-intercepts of a parabola then the equation may be written as $y = a(x - r_1)(x - r_2)$. Furthermore, if we have another point on the parabola, we may plug that point in to determine the value of leading coefficient *a*.

Review material at Section 6.1: Factors and x-Intercepts as needed.

Example 5.2.4

Consider all rectangles with perimeter 12. We develop a quadratic relationship between the base x and the area y as follows.

a) Draw all possible rectangles with perimeter 12 with whole number dimensions.

b) Fill in the areas in the table below.

Base (x)	0	1	2	3	4	5	6
Area (y)	0						0

c) Sketch and connect the points

d) Write an equation that expresses the base x as a function of the area y.

e) What is the independent variable? What is the dependent variable? Also state what each of these variables represent in terms of the rectangles.

f) What is the domain (including all points on the graph)? What is the range (including all points on the graph)?

g) For what x-values is the function increasing? For what x-values is the function decreasing? What does this tell us about rectangles with perimeter 12?

h) What is the highest point on the graph and what does it tell us about rectangles with perimeter 12?

Example 5.2.5

The following table represents a quadratic function. Use the information in the table to answer the questions below.

x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
y	-54	-24	0	18	30	36	36	30	18	0	-24	-54

a) What are the x-intercepts?

b) What is the line/axis of symmetry? Explain how you determined this.

c) Write an equation for this quadratic function.

d) Does the graph open up or down? Explain how you determined this.

e) What is the vertex? Explain how you determined this.

Example 5.2.6

The table below represents a quadratic function. Use this information to answer the questions below. Do not sketch until you get to part (e).

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-175	-120	-75	-40	-15	0	5	0	-15	-40

a) What are the x-intercepts?

b) What is the axis of symmetry? Explain.

c) What is the vertex? Explain.

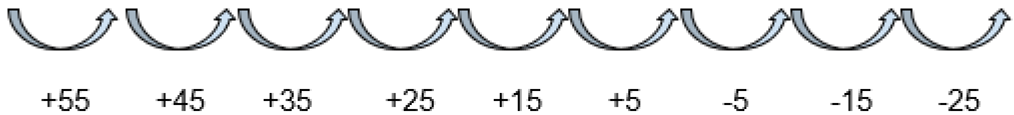
d) Does the graph open up or down? Explain.

e) Sketch the graph and find the equation.

Concavity of Quadratics

Recall the table from the previous example

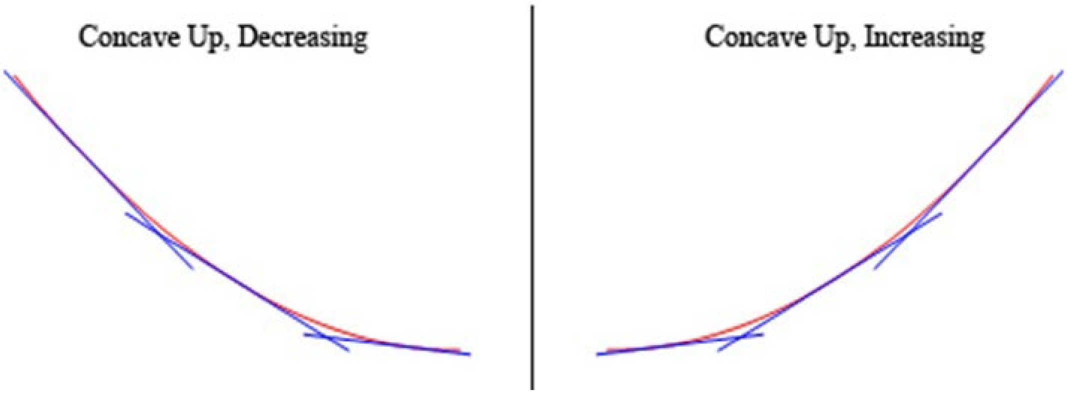
x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-175	-120	-75	-40	-15	0	5	0	-15	-40



Observe that before the vertex the y-values are increasing by less each time. After the vertex, the y-values are decreasing by more each time (as we go from left to right). This change in the rate of increase or decrease of the y-values is what gives the parabola its curved shape (also known as “concavity” which we discussed in chapter 3). With linear functions, the y-values increase or decrease by the same amount for each x-value increment (constant slope) but with parabolas, the slope (rate of increase/decrease) changes. In general (looking from left to right)

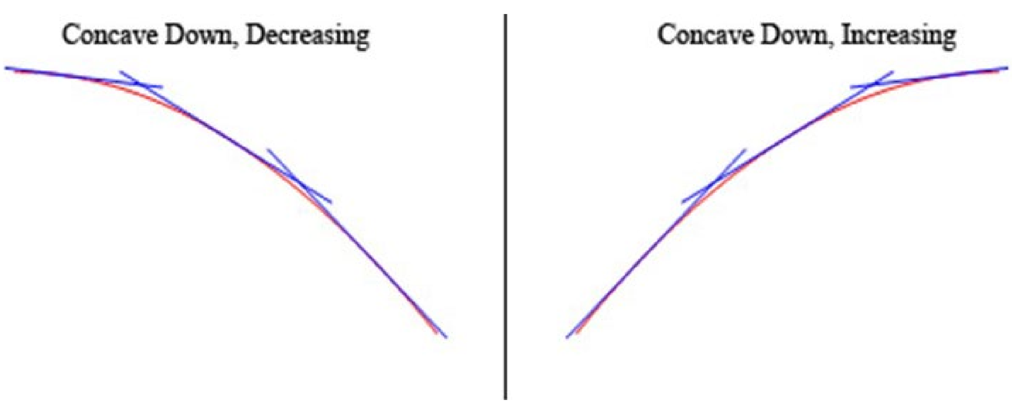
Slopes get less steep

Slopes get more steep



Slopes get more steep

Slopes get less steep



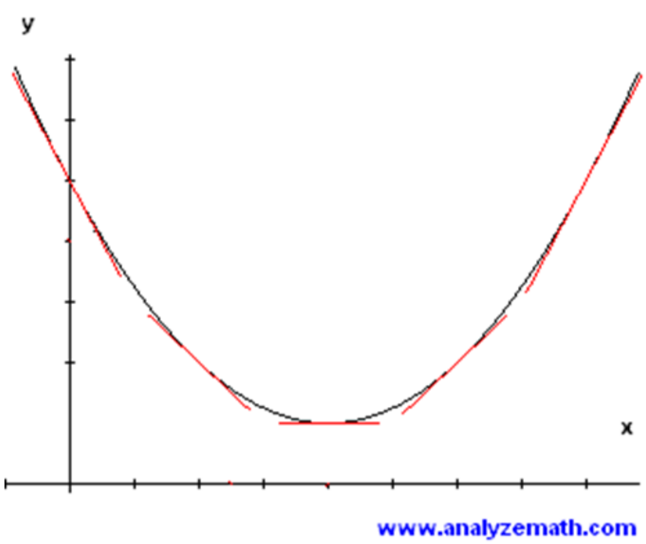
Take another look at the table from the previous example. **Do you notice anything else special about how the y-values are changing?** Write what you see below.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-175	-120	-75	-40	-15	0	5	0	-15	-40

+55 +45 +35 +25 +15 +5 -5 -15 -25

Example 5.2.7

Consider the parabola below.



a) Explain what is happening with the slopes (as we move from left to right).

b) What does this say about how and where the function is increasing/decreasing?

Example 5.2.8

Find the equation of the quadratic function represented in the table below using the form $y = ax^2 + bx + c$ as follows. First, use the y-intercept to find *c*. Then, plug in two different points to get two equations with unknowns *a* and *b*. Solve this system of equations as you learned in chapter 4 (using elimination or substitution).

x	-5	-4	-3	-2	-1	0	1	2	3
y	49	25	9	1	1	9	25	49	81

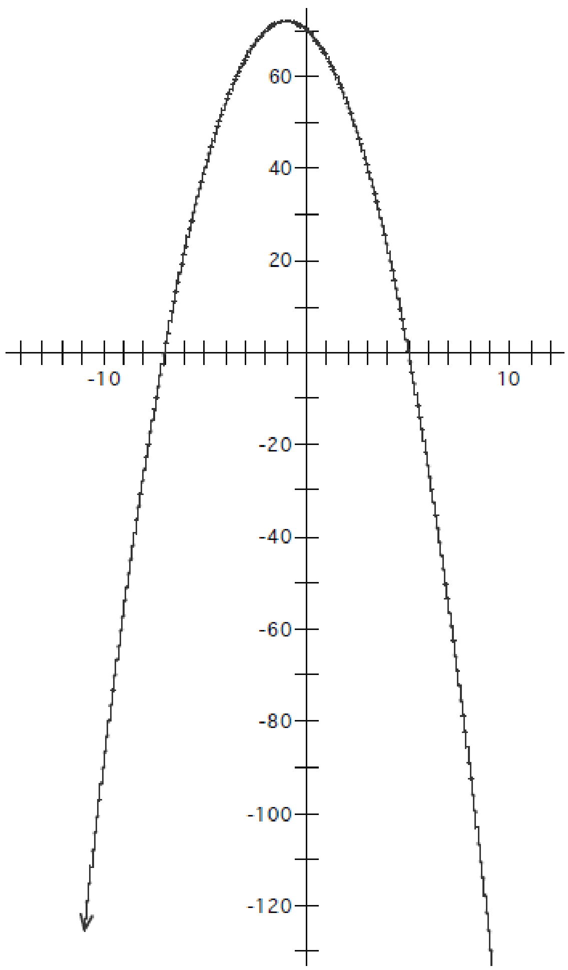
Example 5.2.9

Find the equation of the quadratic function represented in the table below using the form $y = a(x-h)^2 + k$ as follows. Find the vertex (h,k) in the table. Then plug in another point to find *a*.

x	-5	-4	-3	-2	-1	0	1	2	3
y	33	22	13	6	1	-2	-3	-2	1

Example 5.2.10

Consider the parabola below.



a) What are the x-intercepts? Mark them on the graph.

b) What is the line of symmetry? Sketch it with a dotted line on the graph.

c) What is the vertex? Mark it on the graph.

d) What is the equation for the function? Show your work.

Example 5.2.11

Sketch the graph of the quadratic function $y = x^2 - 2x - 8$. Find and show x and y intercepts and vertex.

Example 5.2.12

Huang is shooting a basketball. The path of the ball is parabolic in shape. The ball reaches a maximum height of 11.5 feet when it is 10 feet from where Huang is standing. The basketball net/hoop is 10 feet high and the ball hits it.

a) Find an equation for the height of the ball as a function of its horizontal distance from Huang. First draw a picture of the ball path.

b) Use your answer to part (a) to determine Huang's horizontal distance from the hoop.

5.2 Exercises

1) Consider all rectangles with perimeter 24. Proceed as we did in Example 5.2.4 and develop a functional relationship between the **base** x and the **area** y as follows.

   a) Draw all possible rectangles with perimeter 24 with whole number dimensions.

b) Fill in the table below based on your results from part (a)

Base (x)	0	1	2	3	4	5	6	7	8	9	10	11	12
Area (y)	0												0

c) Sketch and connect the points from your table.

d) Write an equation that expresses the base x as a function of the area y.

e) What is the highest point on the graph and what does it tell us about rectangles with perimeter 24?

2) Find the equation of each of the following quadratic functions represented in the tables below using one of the methods we learned. Which method you use depends on the information you have (e.g. vertex, intercepts).

a)

x	-5	-4	-3	-2	-1	0	1	2	3
y	2.5	-1	-3.5	-5	-5.5	-5	-3.5	-1	2.5

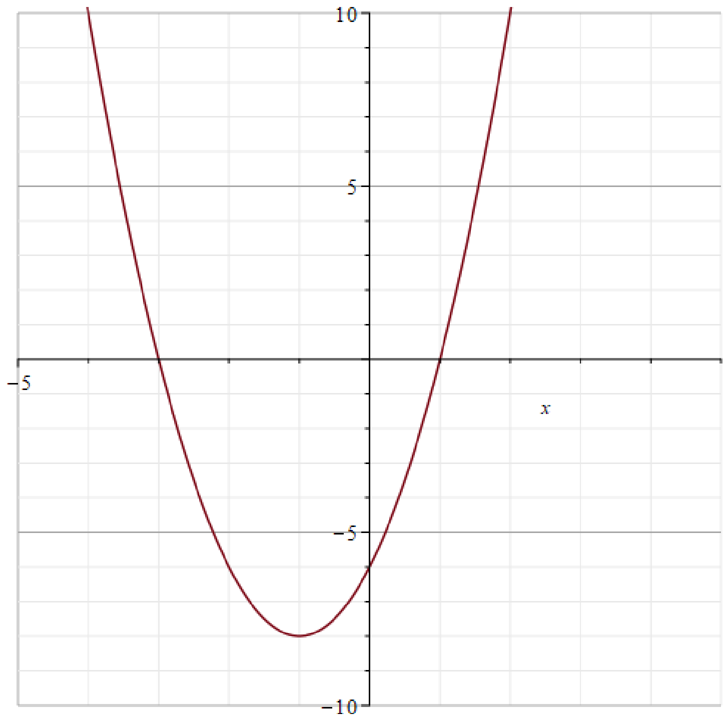
b)

x	-2	-1	0	1	2	3
y	14	3	-2	-1	6	19

c)

x	-2	-1	0	1	2	3
y	-1	-4	-5	-4	-1	4

3) Answer the following questions for the parabola below.



a) What are the x-intercepts? Mark them on the graph.

b) What is the line of symmetry? Sketch it with a dotted line on the graph.

c) What is the vertex? Mark it on the graph.

d) What is the equation for the function? Show your work.

4) Graph the following quadratic functions. Be sure to find and show x and y intercepts and vertex for each.

a) $y = x^2 + 4x - 3$

b) $y = x^2 + 5x + 4$

c) $y = 4x^2 - 16x$

5) Which of the following relationships are quadratic? Justify your answer.

   a) The relationship between the radius of a circle and its area.

   b) The relationship between the length of a square and its perimeter.

   c) The relationship between the radius of a sphere and its volume.

   d) One soccer team has *n* players and the other has *(n - 1)* players. Each player from one team shakes every player's hand on the opposing team. Consider the relationship between *n* and the number of handshakes.

Chapter 5 Wrap Up

In this chapter we studied quadratic functions, their properties, and applications thereof. Quadratic functions are a well known type of nonlinear function, but there are other nonlinear functions that are very important in mathematical modeling. In particular, exponential functions are nonlinear and widely used in various real world applications. Go to the following link to learn more about exponential functions: https://yoshiwarabooks.org/mfg/chap4.html